INTERNATIONAL MATHEMATICS

TOURNAMENT OF TOWNS

JUNIOR PAPER: YEARS 8,9,10

Tournament 39, Northern Spring 2018 (O Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. Six rooks are placed on a 6×6 board, so that none is under attack. Then, each unoccupied square is coloured red or blue according to the following rule:

> If all the rooks that attack that square are located at the same distance from the square, then it is coloured red. Otherwise it is coloured blue.

Is it possible that after colouring all the unoccupied squares, they are

(a) red? (1 point)

(b) blue? (2 points)

2. Let K be a point on the hypotenuse AB of a right-angled triangle ABC, and let L be a point on the side AC such that $AK = AC$ and $BK = LC$ respectively. Let M be the point of intersection of the line segments BL and CK. Prove that triangle CLM is isosceles. (4 points)

3. An integer has been written in each square of a 4×4 table. The sums of the numbers in each column and each row of the table are the same. Seven of the numbers in the table are known, while the rest have been lost (see diagram below).

1	?	?	2
?	4	5	?
?	6	7	?
3	?	?	?

Continued over the page

Is it possible having only the above information to restore

(a) at least one of the lost numbers? (2 points)

(b) at least two of the lost numbers? (2 points)

4. Three positive integers are given such that each of them is divisible by the greatest common divisor of the other two numbers, and the least common multiple of any two is divisible by the third number. Are these three numbers necessarily equal to each other? (4 points)

5. Thirty points have been chosen in the plane so that no three lie on the same line. Then 7 red lines are drawn so that they do not contain any of the chosen points. Is it possible that each line segment connecting two chosen points crosses at least one red line? (5 points)

O Level Junior Paper Solutions

Edited by Oleksiy Yevdokimov and Greg Gamble

1. Since no rook directly attacks another rook, there is at most one rook in each row. Thus, since there are 6 rows and 6 rooks, there is exactly one rook in each row. Similarly, there is exactly one rook in each column.

   Suppose the square S in row i and column j is unoccupied. Then S is attacked by the rook R_1 in row i, whose column is n, say, and whose distance from S is $|j-n|$. Also S is attacked by the rook R_2 in column j, whose row is m, say, and whose distance from S is $|i-m|$.

   Now R_1 and R_2 lie on a diagonal if and only if $|i-m| = |j-n|$.

   This observation allows us to answer both (a) and (b), affirmatively.

   (a) Yes, it is possible, that all unoccupied squares can be coloured red. An unoccupied square is attacked by two rooks at the same distance from the square if they lie on a diagonal. Thus if all the rooks lie on the same diagonal, then each unoccupied square is attacked by two rooks that are at the same distance from the square. In order for all the rooks to lie on the same diagonal, they must lie on a main diagonal. One configuration is pictured below.

					R
				R	
			R		
		R			
	R				
R					

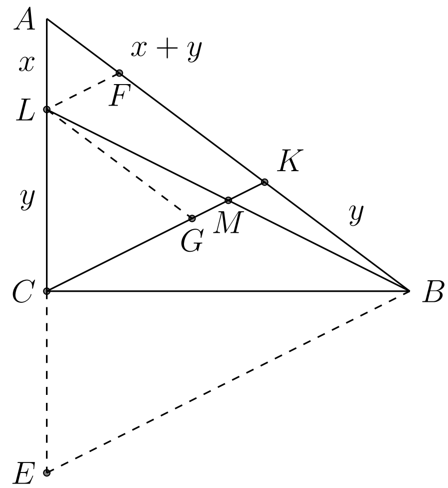
   (b) Yes, it is possible, that all unoccupied squares are coloured blue. By the criterion, we must find a configuration where no rook shares a diagonal with another rook (this is equivalent to positioning 6 queens so that they don't attack one another, instead of 6 rooks). A configuration with this property is shown below.

		R			
					R
	R				
				R	
R					
			R		

Note. Other locations of the rooks can be obtained by rotations or refections of the board.

2. **Solution 1.** We claim that M is the midpoint of LB. Indeed, choose point F on AK such that $AF = AL$. According to Thales' Intercept Theorem[1] FL and KC are parallel to each other and $FK = LC = BK$. Hence, KM is a middle line of triangle BFL and, therefore, $BM = LM$.

   Thus, in right-angled triangle BCL point M is equidistant from all three vertices B, C and L. So, triangle CLM is isosceles.



   Solution 2. We claim that M is the midpoint of LB. Indeed, through point L construct a line parallel to AB that meets CK at point G. Hence, triangles CAK and CLG are similar and since triangle CAK is isosceles, so is triangle CLG. Therefore, $LG = LC = BK$. Since LG and BK are parallel each other, we conclude that $LGBK$ is a parallelogram. Hence, M is the midpoint of LB as a point where the diagonals of parallelogram $LGBK$ meet.

   Thus, in right-angled triangle BCL point M is equidistant from all three vertices B, C and L. So, triangle CLM is isosceles.

   Solution 3. We claim that M is the midpoint of LB. Indeed, place masses $x = AL$, $y = LC$ and $x + y$ at points C, A and B respectively. Then, L is the centre of mass of the points A and C, and K is the centre of mass of the points A and B. Thus, the common centre of mass is located at the intersection of BL and CK, which is M. Considering A and C we get point L has mass $x + y$. Since points L and B have equal masses, M is the midpoint of LB.

   Thus, in right-angled triangle BCL point M is equidistant from all three vertices B, C and L. So, triangle CLM is isosceles.

   Solution 4. Through point B construct a line parallel to CK that meets AC at point E. Hence, triangles CAK and EAB are similar and since triangle CAK is isosceles, so is triangle EAB. Therefore, $EC = BK = LC$. Thus, altitude BC is also a median in triangle ELB, which means triangle ELB is isosceles. Since triangles ELB and CLM are similar and since triangle ELB is isosceles, so is triangle CLM, as required.

[1] There are two theorems attributed to Thales. We refer to:

Thales' Intercept Theorem. If a straight line is drawn parallel to one of the sides of a triangle, then it cuts the other sides of the triangle, in the same proportion.

Note. In the spirit of **Solution 4.**, a line parallel to AC can be constructed through point B that meets line CK at point P. Then, after proving that $BCLP$ is a rectangle, it follows that CLM is an isosceles triangle.

3. (a) Yes, it is possible to restore one of the lost numbers. Let the rightmost bottom unknown number be x. Then, the sum of the numbers in the two middle columns is equal to the sum of the numbers in the top and bottom rows. So, after cancelling the common elements in th columns and rows, we get

$$4+5+6+7=1+2+3+x,$$

from which we obtain $x=16$.

(b) No, it is not possible to restore more than one of the lost numbers. Indeed, assume we have a suitable set of eight unknown numbers. If we add 1 to each of those eight unknowns, the sum of any row/column will increase by 2, which means that the property that the sums of the numbers in each column and each row of the table are equal, is preserved. Thus, since there are multiple solutions, for any of the 8 unknown numbers, none of them can be restored.

Note. Suitable sets exist, for example,

1	11	11	2
11	4	5	5
10	6	7	2
3	4	2	16

and

1	13	13	2
12	4	5	8
13	6	7	3
3	6	4	16

.

It is not possible to restore any other number than a number in the right bottom corner even if the sum of the numbers in each column and each row is known. Indeed, the following table can be added to any table (adding corresponding entries together like in matrix addition) without changing the totals in each column and each row, where the newly-obtained table will satisfy all required conditions.

0	1	-1	0
-1	0	0	1
1	0	0	-1
0	-1	1	0

4. **Solution 1.** Yes, these three numbers are necessarily equal to each other. Let the three numbers be x, y and z. We claim that any prime number p has the same power in prime factorisation of x, y and z. Indeed, let p has power α in prime factorisation of x, power β in prime factorisation of y and power γ in prime factorisation of z. Without loss of generality we assume that $\alpha \leq \beta \leq \gamma$. Since x is divisible by the greatest common divisor of y and z, we get $\alpha \geq \beta$. Since the least common multiple of x and y is divisible by z, we get $\beta \geq \gamma$. Hence, $\alpha = \beta = \gamma$ and the conclusion follows.

Solution 2. Yes, these three numbers are necessarily equal to each other. Let d be the greatest common divisor of the three numbers. Then, these three numbers can be represented as xd, yd and zd, where x, y and z are coprime. We claim that $x = y = z = 1$. We show the case of x. The cases of y and z can be done similarly. Indeed, suppose a prime number p divides x. Denote the least common multiple of two integers a and b by $[a, b]$ and the greatest common divisor by (a, b). Since $[y, z]d$ is divisible by xd, $[y, z]d$ is divisible by pd. Hence, either y or x is divisible by p. Without loss of generality assume y is divisible by p. Since zd is divisible by $(x, y)d$, zd is divisible by pd. Thus, x, y and z are divisible by p which leads to a contradiction. Hence, $x = 1$. Similarly, $y = z = 1$, and the conclusion follows.

5. No, it is not possible. We claim that for natural numbers n,

   $P(n)$: n straight lines partition the plane into not more than $1+n(n+1)/2$ regions.

   Our proof is by induction.

   Since a single line partitions the plane into 2 pieces, and $1 + 1 \cdot (1 + 1)/2 = 2$, it follows that $P(1)$ holds.

   Now suppose that $P(n - 1)$ holds. Then by the the induction hypothesis, $n - 1$ straight lines partition the plane into not more than $1 + n(n - 1)/2$ regions. A further line is cut into n parts by the points of intersections with the first $n - 1$ lines. Each part divides a corresponding region into two regions. Hence, n lines partition the plane into n more regions than $n - 1$ lines do, i.e. n lines partition the plane into not more than $1 + n(n - 1)/2 + n = 1 + n(n + 1)/2$ regions. Thus, $P(n)$ holds, if $P(n - 1)$ does.

   So the induction is complete, and hence $P(n)$ holds for all natural numbers n.

   Thus, in particular, 7 red lines divides the plane into not more than 29 regions. Hence, by the Pigeon-Hole Principle, at least one of these at most 29 regions must contain at least 2 of the 30 points, which means the line segment connecting those two points does not cross any red line.

   Note. The requirement that no three points lie on the same line is unnecessary.